

The University of Chicago

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OF MODEL STELLAR
ATMOSPHERES

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

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RALPH E. WILLIAMSON

THE NEGATIVE HYDROGEN ION AND ITS ABSORPTION COEFFICIENT

RALPH E. WILLIAMSON

ABSTRACT

A six-parameter wave-function for the negative hydrogen ion has been derived on the basis of the Ritz principle, similar to the wave-function of Hylleraas for the ground state of helium. This new wave-function predicts for the electron affinity a value of 0.0265 atomic units; this value is appreciably different from 0.0253 atomic units currently adopted.

The atomic absorption coefficient of H^- for radiation of different wave lengths has been computed on the basis of the new wave-function. It is found that the absorption coefficient per H^- ion has the maximum value $3.0 \times 10^{-17} \text{ cm}^2$ at 5000 Å (instead of $2.6 \times 10^{-17} \text{ cm}^2$ at 4000 Å, according to the earlier calculations of Massey and Bates). Throughout the red and the infrared the new values for the absorption coefficient are substantially larger than those now in general use.

I. INTRODUCTION

Since Wildt's discovery¹ of the negative hydrogen ions as the principal source of opacity in the solar atmosphere, several investigations² have been carried out to incorporate this new source of opacity in a theoretical analysis of model stellar atmospheres. Most important of these is undoubtedly that of Strömgren,³ whose quantitative analysis shows that now for the first time a theory of the solar atmosphere becomes possible in which no principal discrepancy remains. On the other hand, Wildt's investigations⁴ on the theoretical relation between the color temperature and the effective temperature of stars shows that in this matter there still remains a discordance between observation and the prediction of the theory. Mainly on the strength of this evidence it has been concluded that there might possibly remain further unexplored sources of opacity. However, Wildt's discussion is based on the continuous absorption coefficient $k_\nu(H^-)$ due to H^- as computed by Massey and Bates.⁵ According to their calculations, the absorption coefficient, $k_\nu(H^-)$, attains its maximum at 4000 Å and decreases throughout the visual region. And, as Wildt has pointed out, it is precisely this decrease of the absorption coefficient in the region to the red of 4000 Å which is responsible for the discrepancy noted. A shift of the maximum of the absorption coefficient to the region 5000–6000 Å would remove, or at any rate greatly diminish, the extent of the discordance between theory and observation. Massey and Bates's computation of the absorption coefficient of H^- is based on a "third-order" wave-function for the ground state of H^- due to Bethe⁶ and Hylleraas,⁷ derived on the basis of the Ritz principle. Massey and Bates point out that this wave-function predicts a value of the energy correct to within a fraction of 1 per cent. More precisely, the Bethe-Hylleraas wave-function yields for the energy E of the ground state the value $E = -0.5253$ in Hartree's atomic units, while an extrapolation from other two-electron systems leads to a value $E = -0.5264$.⁸ On the other hand, as Chandrasekhar⁹ has pointed out, what is relevant for the problem of the absorption co-

¹ *Ap. J.*, **89**, 295, 1939.

² Massey and Bates, *Ap. J.*, **91**, 202, 1940; Wildt, *Ap. J.*, **93**, 47, 1941; Strömgren, *Festschrift für Elis Strömgren*, p. 218, Copenhagen, 1940.

³ *Op. cit.*

⁶ *Zs. f. Phys.*, **57**, 815, 1929.

⁴ *Ap. J.*, **93**, 47, 1941.

⁷ *Zs. f. Phys.*, **60**, 624, 1930.

⁵ *Loc. cit.*

⁸ *Ibid.*

⁹ In a lecture (unpublished) given at the Yerkes Observatory.

efficient is not so much the accuracy with which the total energy E is predicted as it is the accuracy with which the electron affinity is predicted. Judged by this criterion, the Bethe-Hylleraas wave-function does not appear sufficiently accurate; the electron affinity which it predicts is quite appreciably less: 0.0253 instead of 0.0264. If this is the error in the eigenvalue, the extent of the inaccuracy of the wave-function is probably much more serious. In any case it appears worth while to set up for H^- a more accurate sixth-order wave-function following the methods developed by Hylleraas for the ground state of helium and to re-evaluate the absorption coefficient on this basis. This is the object of the present paper. As we shall see, the calculations justify our suspicion that the Bethe-Hylleraas wave-function does not provide the accuracy necessary for our purposes (see Fig. 1).

II. DETERMINATION OF THE WAVE-FUNCTION

The Ritz method for solving the wave-equation

$$H\Psi = E\Psi \quad (1)$$

for the lowest eigenstate is based on the fact that the corresponding wave-function Ψ makes E a minimum, where E is defined by

$$E = \frac{\int \bar{\Psi} H \Psi d\tau}{\int |\Psi|^2 d\tau} \quad (2)$$

The integrations in equation (2) are to be extended over the entire configuration space. In practice, a form for the wave-function is assumed which involves certain parameters. And the "best" wave-function of the assumed form is obtained by assigning to these parameters values which will make the right-hand side of equation (2) a minimum. The success of this method clearly depends on how well the initial form for the wave-function is chosen. But Hylleraas has found that the observed value of E for the ground state of helium can be predicted within spectroscopic accuracy, if one assumes for Ψ the form

$$\Psi = e^{-ks/2} [c_0 + c_1(ku) + c_2(kt)^2 + c_3(ks) + c_4(ks)^2 + c_5(ku)^2], \quad (3)$$

where s , t , and u are related to the distances r_1 , r_2 , and r_{12} , of the two electrons from the nucleus and from one another, respectively, according to the relations

$$s = r_1 + r_2, \quad t = -r_1 + r_2, \quad u = r_{12}. \quad (4)$$

In this sixth-order wave-function, k , c_1 , c_2 , c_3 , c_4 , and c_5 are six parameters which are adjusted until E takes the smallest possible value consistent with Ψ being of the form of equation (3). Since this type of wave-function predicts the energy of the ground state of helium with extreme precision,¹⁰ we shall adopt it for H^- also and determine the values of the six parameters which will make E defined by equation (2) a minimum.

A convenient procedure that may be followed in practice is described by Bethe.¹¹ It involves a considerable amount of tedious numerical work, including the evaluation of several sixth-order determinants and the solving of a number of sets of six simultaneous homogeneous linear equations. However, the method was found feasible.

¹⁰ Bethe, in his article on one- and two-electron systems in the *Handbuch der Physik* (XXIV, Part I, 358, Berlin, 1933) compares Hylleraas' value for E with the experimental value.

¹¹ "Quantentheorie," *ibid.*, p. 353.

We give below the results of the computation:

$$\left. \begin{aligned} E &= -0.52646, \\ k &= +1.40239, \quad c_3 = -0.10259, \\ c_1 &= +0.18552, \quad c_4 = +0.00222, \\ c_2 &= +0.03673, \quad c_5 = -0.00455, \end{aligned} \right\} \quad (5)$$

where the constants are expressed in atomic units. We can now write the wave-function for H^- in the form

$$\Psi = e^{-as} (1 + \beta u + \gamma l^2 + \delta s + \epsilon s^2 + \zeta u^2), \quad (6)$$

where

$$\left. \begin{aligned} a &= \frac{k}{2} = +0.70120, & \delta &= c_3 k = -0.14387, \\ \beta &= c_1 k = +0.26017, & \epsilon &= c_4 k^2 = +0.00436, \\ \gamma &= c_2 k^2 = +0.07225, & \zeta &= c_5 k^2 = -0.00896. \end{aligned} \right\} \quad (7)$$

The "normalization factor," N , defined by

$$N^2 \int \Psi^2 d\tau = 1, \quad (8)$$

is related to certain integrals used in the minimizing of E and has the value

$$N = 0.073965. \quad (9)$$

It is of interest to compare the relative accuracy with which the different orders of approximations for the wave-function of H^- predict E and the electron affinity for the negative hydrogen ion. The relevant data are given in Table 1. We see that all wave-

TABLE 1
ELECTRON AFFINITY OF H^- AS PREDICTED
BY DIFFERENT WAVE-FUNCTIONS

	Ψ	E	Electron Affinity
1.....	$e^{-0.688s}$	-0.473	-0.027
2.....	$e^{-0.825s}(1+0.490u)$	.509	+0.009
3.....	$e^{-0.769s}(1+0.318u+0.125l^2)$	.5253	+0.0253
4.....	As in eq. (6)	-.52646	+0.02646
Extrapolation from other two electron systems.....		-0.52642	+0.02642

functions except the first predict the existence of H^- , as they all lead to positive values for the electron affinity. It will be specially noticed that the wave-function which we have just determined actually *increases* the value for the electron affinity over Hylleraas' value extrapolated from the other two-electron systems. It therefore appears likely that our present sixth-order wave-function predicts the value of the electron affinity to within a fraction of 1 per cent. Thus the criticism of the Bethe-Hylleraas wave-function (No. 3 in Table 1) made in section I is not applicable to the present one. It now remains to use this wave-function in the calculation of the absorption coefficient.

III. THE ABSORPTION COEFFICIENT

If a stream of light quanta of frequency ν strikes a slab of matter containing N atoms/cm³, the number of quanta absorbed in penetrating a distance ds of the material is $Nk_\nu ds$. Here k_ν is the effective cross-section of one atom for absorbing the radiation of frequency ν . Thus, when the stream of quanta has traversed a thickness s of the material, its intensity will be reduced by a factor $e^{-Nk_\nu s}$. We desire to evaluate the absorption cross-section k_ν for H^- ions, as a function of the frequency ν of the incident quanta. The absorption results from a transition of the ion to one of its continuous energy states. In particular, if the electron affinity is E_0 , then the velocity v of the photo-electron is given by Einstein's relation

$$\frac{1}{2} m v^2 = h\nu - E_0. \quad (10)$$

The absorption cross-section for such a process is given by the quantum-mechanical formula

$$k_\nu = \frac{32\pi^4 m^2 e^2}{3h^3 c} \nu v |\mu|^2, \quad (11)$$

where μ is the matrix element, corresponding to the transition from the initial state of the hydrogen ion to the final state of a hydrogen atom in the ground state, and an outgoing current of electrons with velocity v and unit density

$$\mu = \int \bar{\Psi}_A (r_1 + r_2) \Psi_B d\tau. \quad (12)$$

In evaluating this matrix element we shall use for the wave-function representing the ground state of the hydrogen ion our sixth-order approximation,

$$\Psi_A = N e^{-\alpha s} (1 + \beta u + \gamma t^2 + \delta s + \epsilon s^2 + \zeta u^2), \quad (13)$$

where the constants $N, \alpha, \beta, \gamma, \delta, \epsilon$, and ζ have the values given in equations (7) and (9). For the wave-function representing the final state we have

$$\Psi_B = \frac{1}{\sqrt{2\pi}} (e^{-r_2 + i\mathbf{p} \cdot \mathbf{r}_1} + e^{-r_1 + i\mathbf{p} \cdot \mathbf{r}_2}), \quad (14)$$

where $\mathbf{p}$ denotes the momentum of the free electron. All quantities in equation (14) are expressed in atomic units. It will be noticed that, as required, Ψ_B has been chosen to correspond to an outgoing current of electrons of unit density and further is symmetrical in the co-ordinates of the two electrons. We can write

$$\Psi_B = \frac{1}{\sqrt{2\pi}} (e^{ikz_1 - r_2} + e^{ikz_2 - r_1}), \quad (k = |\mathbf{p}|), \quad (15)$$

instead of equation (14), if the z -axis of our co-ordinate system is arranged to be in the direction of $\mathbf{p}$.

It is seen that, for Ψ_A and Ψ_B of the forms (13) and (15), μ_x and μ_y vanish identically, on account of the symmetry of the integrand in equation (12), and the only nonvanishing component is μ_z :

$$\mu_z = \frac{N}{\sqrt{2\pi}} \left\{ \int_{r_1=0}^{\infty} \int_{\vartheta_1=0}^{\pi} \int_{\varphi_1=0}^{2\pi} \int_{r_2=0}^{\infty} \int_{\vartheta_2=0}^{\pi} \int_{\varphi_2=0}^{2\pi} e^{-\alpha(r_1+r_2)} [1 + \beta r_{12} + \gamma (r_2 - r_1)^2 + \delta (r_1 + r_2) + \epsilon (r_1 + r_2)^2 + \zeta r_{12}^2] (z_1 + z_2) (e^{ikz_1 - r_2} + e^{ikz_2 - r_1}) \times r_1^2 r_2^2 d r_1 d r_2 \sin \vartheta_1 \sin \vartheta_2 d \vartheta_1 d \vartheta_2 d \varphi_1 d \varphi_2 \right\} \quad (16)$$

The angle integrations indicated in equation (16) are readily carried out when all the functions involving the polar angles ϑ_1 and ϑ_2 are expressed in terms of Legendre polynomials. This can be conveniently done by using the following series expansions.

The expansion for the plane wave in terms of half-integral Bessel functions is well known.¹² We have

$$\left. \begin{aligned} e^{ikz} &= e^{ikr \cos \vartheta} \\ &= \left(\frac{\pi}{2kr} \right)^{1/2} \sum_{n=0}^{\infty} i^n (2n+1) J_{n+1/2}(kr) P_n(\cos \vartheta), \end{aligned} \right\} \quad (17)$$

where $J_{n+1/2}(x)$ denotes the Bessel function of order $n+\frac{1}{2}$, and $P_n(x)$ the Legendre polynomial of order n . A series expansion for r_{12} in terms of Legendre polynomials of the angle Θ between r_1 and r_2 may be derived:¹³

$$\left. \begin{aligned} r_{12} &= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos \Theta} \\ &= \hat{r} \sum_{\nu=0}^{\infty} \left(\frac{x^2}{2\nu+3} - \frac{1}{2\nu-1} \right) x^\nu P_\nu(\cos \Theta), \end{aligned} \right\} \quad (18)$$

where

$$\cos \Theta = \cos \vartheta_1 \cos \vartheta_2 + \sin \vartheta_1 \sin \vartheta_2 \cos(\varphi_1 - \varphi_2) \quad (19)$$

and

$$\left. \begin{aligned} \hat{r} &= r_1, & x &= \frac{r_2}{r_1}, & \text{if } r_1 > r_2; \\ \hat{r} &= r_2, & x &= \frac{r_1}{r_2}, & \text{if } r_1 < r_2. \end{aligned} \right\} \quad (20)$$

To make use of the series (18), we must further express $P_n(\cos \Theta)$ in terms of associated Legendre polynomials in $\cos \vartheta_1$ and $\cos \vartheta_2$, by using the addition theorem,¹⁴

$$P_\nu(\cos \Theta) = \sum_{\mu=-\nu}^{\nu} \frac{(\nu - |\mu|)!}{(\nu + |\mu|)!} P_\nu^\mu(\cos \vartheta_1) P_\nu^\mu(\cos \vartheta_2) e^{i\mu(\varphi_1 - \varphi_2)}. \quad (21)$$

We can now express equation (16) in the form

$$\mu_z = \frac{N}{\sqrt{2\pi}} (A + B\beta + \Gamma\gamma + \Delta\delta + E\epsilon + Z\zeta), \quad (22)$$

¹² See, e.g., Mott and Massey, *The Theory of Atomic Collisions*, p. 20, Oxford, 1933.

¹³ Jen (*Phys. Rev.*, **43**, 540, 1933) gives relation (18). It can be readily obtained by assuming a series of the form

$$r_{12} = \hat{r} \sum_{\nu=0}^{\infty} a_\nu x^\nu,$$

differentiating it with respect to $\cos \Theta$, and comparing it with the series

$$\frac{1}{r_{12}} = \hat{r} \sum_{n=0}^{\infty} P_n(\cos \Theta) x^n.$$

¹⁴ Macmillan, *Theory of the Potential*, p. 376, New York, 1930.

where we have written

$$\left. \begin{aligned} A &= \iint e^{-a(r_1+r_2)} (r_1 \cos \vartheta_1 + r_2 \cos \vartheta_2) (e^{ikz_1-r_2} + e^{ikz_2-r_1}) dr_1 dr_2, \\ B &= \iint e^{-a(r_1+r_2)} r_{12} (r_1 \cos \vartheta_1 + r_2 \cos \vartheta_2) (e^{ikz_1-r_2} + e^{ikz_2-r_1}) dr_1 dr_2, \\ \Gamma &= \iint e^{-a(r_1+r_2)} (r_2 - r_1)^2 (r_1 \cos \vartheta_1 + r_2 \cos \vartheta_2) (e^{ikz_1-r_2} + e^{ikz_2-r_1}) dr_1 dr_2, \\ \Delta &= \iint e^{-a(r_1+r_2)} (r_1 + r_2) (r_1 \cos \vartheta_1 + r_2 \cos \vartheta_2) (e^{ikz_1-r_2} + e^{ikz_2-r_1}) dr_1 dr_2, \\ E &= \iint e^{-a(r_1+r_2)} (r_1 + r_2)^2 (r_1 \cos \vartheta_1 + r_2 \cos \vartheta_2) (e^{ikz_1-r_2} + e^{ikz_2-r_1}) dr_1 dr_2, \\ Z &= \iint e^{-a(r_1+r_2)} r_{12}^2 (r_1 \cos \vartheta_1 + r_2 \cos \vartheta_2) (e^{ikz_1-r_2} + e^{ikz_2-r_1}) dr_1 dr_2. \end{aligned} \right\} \quad (23)$$

All the foregoing integrals except B and Z are of the same type:

$$\left. \begin{aligned} I &= \iint f(r_1, r_2) (r_1 \cos \vartheta_1 + r_2 \cos \vartheta_2) (e^{ikz_1-r_2} + e^{ikz_2-r_1}) dr_1 dr_2, \\ f(r_1, r_2) &= f(r_2, r_1). \end{aligned} \right\} \quad (24)$$

By using equations (17) and (21) we can readily integrate equation (24) over the angles φ_1 , φ_2 , ϑ_1 , and ϑ_2 , making use of the orthogonality properties of the Legendre polynomials. We obtain

$$I = 32\pi^2 i \left(\frac{\pi}{2k} \right)^{1/2} \int_0^\infty \int_0^\infty f(r_1, r_2) e^{-r_2} r_1^{5/2} r_2^2 J_{3/2}(kr_1) dr_1 dr_2. \quad (25)$$

Accordingly, we have

$$\left. \begin{aligned} A &= 2q \int_0^\infty \int_0^\infty e^{-(a+1)r_2 - ar_1} r_1^{5/2} r_2^2 J_{3/2}(kr_1) dr_1 dr_2, \\ \Gamma &= 2q \int_0^\infty \int_0^\infty e^{-(a+1)r_2 - ar_1} (r_2 - r_1)^2 r_1^{5/2} r_2^2 J_{3/2}(kr_1) dr_1 dr_2, \\ \Delta &= 2q \int_0^\infty \int_0^\infty e^{-(a+1)r_2 - ar_1} (r_1 + r_2) r_1^{5/2} r_2^2 J_{3/2}(kr_1) dr_1 dr_2, \\ E &= 2q \int_0^\infty \int_0^\infty e^{-(a+1)r_2 - ar_1} (r_1 + r_2)^2 r_1^{5/2} r_2^2 J_{3/2}(kr_1) dr_1 dr_2, \end{aligned} \right\} \quad (26)$$

where for the sake of brevity we have written

$$q = 16\pi^2 i \left(\frac{\pi}{2k} \right)^{1/2}. \quad (27)$$

To evaluate Z we split it into two parts:

$$Z = Z_1 - 2Z_2, \quad (28)$$

where

$$\left. \begin{aligned} Z_1 &= \iint e^{-a(r_1+r_2)} (r_1^2 + r_2^2) (r_1 \cos \vartheta_1 + r_2 \cos \vartheta_2) (e^{ikz_1-r_2} + e^{ikz_2-r_1}) dr_1 dr_2, \\ Z_2 &= \iint e^{-a(r_1+r_2)} r_1 r_2 \cos \Theta (r_1 \cos \vartheta_1 + r_2 \cos \vartheta_2) (e^{ikz_1-r_2} + e^{ikz_2-r_1}) dr_1 dr_2. \end{aligned} \right\} \quad (29)$$

Z_1 is of the form of equation (24) and hence (cf. eq. [25])

$$Z_1 = 2q \int_0^\infty \int_0^\infty e^{-(a+1)r_2 - ar_1} (r_1^2 + r_2^2) r_1^{5/2} r_2^2 J_{3/2}(kr_1) dr_1 dr_2. \quad (30)$$

Remembering that $\cos \Theta = P_1(\cos \Theta)$ and applying equation (21), we effect the angle integrations in Z_2 as we did in equation (25). We obtain

$$Z_2 = \frac{2}{3} q \int_0^\infty \int_0^\infty e^{-(a+1)r_2 - ar_1} r_1^{5/2} r_2^4 J_{3/2}(kr_1) dr_1 dr_2. \quad (31)$$

The angle integrations for B are carried out, after using equations (18) and (21), in the same way as equations (25) and (31) were obtained:

$$\left. \begin{aligned} B &= 2q \int_0^\infty \int_0^\infty e^{-(a+1)r_2 - ar_1} \hat{r} \left(\frac{x^2}{3} + 1 \right) r_1^{5/2} r_2^2 J_{3/2}(kr_1) dr_1 dr_2 \\ &+ \frac{2}{3} q \int_0^\infty \int_0^\infty e^{-(a+1)r_2 - ar_1} \hat{r} \left(\frac{x^3}{5} - x \right) r_1^{3/2} r_2^3 J_{3/2}(kr_1) dr_1 dr_2. \end{aligned} \right\} \quad (32)$$

This completes the angle integrations involved in obtaining the matrix element.

The integrals to be evaluated in A , Γ , Δ , E , and Z are seen to be

$$A = 2q \int_0^\infty e^{-ar_1} r_1^{5/2} J_{3/2}(kr_1) dr_1 \int_0^\infty e^{-(a+1)r_2} r_2^2 dr_2, \quad (33)$$

and

$$\Gamma = 2q (\Gamma_1 - 2\Gamma_2 + \Gamma_3), \quad (34)$$

where

$$\left. \begin{aligned} \Gamma_1 &= \int_0^\infty e^{-ar_1} r_1^{9/2} J_{3/2}(kr_1) dr_1 \int_0^\infty e^{-(a+1)r_2} r_2^2 dr_2, \\ \Gamma_2 &= \int_0^\infty e^{-ar_1} r_1^{7/2} J_{3/2}(kr_1) dr_1 \int_0^\infty e^{-(a+1)r_2} r_2^3 dr_2, \\ \Gamma_3 &= \int_0^\infty e^{-ar_1} r_1^{5/2} J_{3/2}(kr_1) dr_1 \int_0^\infty e^{-(a+1)r_2} r_2^4 dr_2, \end{aligned} \right\} \quad (35)$$

and

$$\Delta = 2q (\Delta_1 + \Delta_2), \quad (36)$$

where

$$\left. \begin{aligned} \Delta_1 &= \int_0^\infty e^{-ar_1} r_1^{7/2} J_{3/2}(kr_1) dr_1 \int_0^\infty e^{-(a+1)r_2} r_2^2 dr_2, \\ \Delta_2 &= \int_0^\infty e^{-ar_1} r_1^{5/2} J_{3/2}(kr_1) dr_1 \int_0^\infty e^{-(a+1)r_2} r_2^3 dr_2. \end{aligned} \right\} \quad (37)$$

Further

$$E = 2q(\Gamma_1 + 2\Gamma_2 + \Gamma_3), \quad Z_1 = 2q(\Gamma_1 + \Gamma_3), \quad Z_2 = \frac{2}{3}q\Gamma_3. \quad (38)$$

In order to express the integrals for B in terms of r_1 and r_2 , we note that

$$\left. \begin{aligned} & \int_{r_1=0}^{\infty} \int_{r_2=0}^{\infty} f(r_1, r_2, x, \hat{r}) dr_2 dr_1 \\ &= \int_{r_1=0}^{\infty} \int_{r_2=0}^{r_1} f\left(r_1, r_2, \frac{r_2}{r_1}, r_1\right) dr_2 dr_1 + \int_{r_1=0}^{\infty} \int_{r_2=r_1}^{\infty} f\left(r_1, r_2, \frac{r_1}{r_2}, r_2\right) dr_2 dr_1, \end{aligned} \right\} \quad (39)$$

in accordance with equations (20). It is seen that we can write

$$B = 2q\left(\frac{1}{3}B_1 + B_2 + \frac{1}{15}B_3 - \frac{1}{3}B_4\right), \quad (40)$$

where

$$\left. \begin{aligned} B_1 &= \int_{r_1=0}^{\infty} e^{-ar_1} r_1^{3/2} J_{3/2}(kr_1) \int_{r_2=0}^{r_1} e^{-(a+1)r_2} r_2^4 dr_2 dr_1 \\ &\quad + \int_{r_1=0}^{\infty} e^{-ar_1} r_1^{9/2} J_{3/2}(kr_1) \int_{r_2=r_1}^{\infty} e^{-(a+1)r_2} r_2^4 dr_2 dr_1, \\ B_2 &= \int_{r_1=0}^{\infty} e^{-ar_1} r_1^{7/2} J_{3/2}(kr_1) \int_{r_2=0}^{r_1} e^{-(a+1)r_2} r_2^2 dr_2 dr_1 \\ &\quad + \int_{r_1=0}^{\infty} e^{-ar_1} r_1^{5/2} J_{3/2}(kr_1) \int_{r_2=r_1}^{\infty} e^{-(a+1)r_2} r_2^3 dr_2 dr_1, \\ B_3 &= \int_{r_1=0}^{\infty} e^{-ar_1} r_1^{-1/2} J_{3/2}(kr_1) \int_{r_2=0}^{r_1} e^{-(a+1)r_2} r_2^6 dr_2 dr_1 \\ &\quad + \int_{r_1=0}^{\infty} e^{-ar_1} r_1^{9/2} J_{3/2}(kr_1) \int_{r_2=r_1}^{\infty} e^{-(a+1)r_2} r_2 dr_2 dr_1, \\ B_4 &= \int_{r_1=0}^{\infty} e^{-ar_1} r_1^{3/2} J_{3/2}(kr_1) \int_{r_2=0}^{r_1} e^{-(a+1)r_2} r_2^4 dr_2 dr_1 \\ &\quad + \int_{r_1=0}^{\infty} e^{-ar_1} r_1^{5/2} J_{3/2}(kr_1) \int_{r_2=r_1}^{\infty} e^{-(a+1)r_2} r_2^3 dr_2 dr_1. \end{aligned} \right\} \quad (41)$$

The integrations over r_2 are readily performed, using the elementary recursion formula

$$\left. \begin{aligned} K_j &= \int e^{-qr} r^j dr \\ &= -\frac{1}{q} \{ e^{-qr} r^j - jK_{j-1} \}. \end{aligned} \right\} \quad (42)$$

After the integrations over r_2 have been carried out, we see that the remaining integrals are of the general type

$$J_m(q) = \int_0^{\infty} e^{-qr} r^{m+1/2} J_{3/2}(kr) dr. \quad (43)$$

Thus,

$$\left. \begin{aligned}
 A &= \frac{4q}{(\alpha+1)^3} \mathcal{J}_2(\alpha), & \Delta_1 &= \frac{2}{(\alpha+1)^3} \mathcal{J}_3(\alpha), & \Delta_2 &= \frac{6}{(\alpha+1)^4} \mathcal{J}_2(\alpha), \\
 \Gamma_1 &= \frac{2}{(\alpha+1)^3} \mathcal{J}_4(\alpha), & \Gamma_2 &= \frac{6}{(\alpha+1)^4} \mathcal{J}_3(\alpha), & \Gamma_3 &= \frac{24}{(\alpha+1)^5} \mathcal{J}_2(\alpha), \\
 B_1 &= \frac{24}{(\alpha+1)^5} \mathcal{J}_1(\alpha) - \frac{24}{(\alpha+1)^5} \mathcal{J}_1(2\alpha+1) - \frac{24}{(\alpha+1)^4} \mathcal{J}_2(2\alpha+1) \\
 &\quad - \frac{12}{(\alpha+1)^3} \mathcal{J}_3(2\alpha+1) - \frac{3}{(\alpha+1)^2} \mathcal{J}_4(2\alpha+1), \\
 B_2 &= \frac{2}{(\alpha+1)^3} \mathcal{J}_3(\alpha) + \frac{6}{(\alpha+1)^4} \mathcal{J}_2(2\alpha+1) + \frac{4}{(\alpha+1)^3} \mathcal{J}_3(2\alpha+1) \\
 &\quad + \frac{1}{(\alpha+1)^2} \mathcal{J}_4(2\alpha+1), \\
 B_3 &= \frac{720}{(\alpha+1)^7} \mathcal{J}_{-1}(\alpha) - \frac{720}{(\alpha+1)^7} \mathcal{J}_{-1}(2\alpha+1) - \frac{720}{(\alpha+1)^6} \mathcal{J}_0(2\alpha+1) \\
 &\quad - \frac{360}{(\alpha+1)^5} \mathcal{J}_1(2\alpha+1) - \frac{120}{(\alpha+1)^4} \mathcal{J}_2(2\alpha+1) - \frac{30}{(\alpha+1)^3} \mathcal{J}_3(2\alpha+1) \\
 &\quad - \frac{5}{(\alpha+1)^2} \mathcal{J}_4(2\alpha+1), \\
 B_4 &= \frac{24}{(\alpha+1)^5} \mathcal{J}_1(\alpha) - \frac{24}{(\alpha+1)^5} \mathcal{J}_1(2\alpha+1) - \frac{18}{(\alpha+1)^4} \mathcal{J}_2(2\alpha+1) \\
 &\quad - \frac{6}{(\alpha+1)^3} \mathcal{J}_3(2\alpha+1) - \frac{1}{(\alpha+1)^2} \mathcal{J}_4(2\alpha+1).
 \end{aligned} \right\} \quad (44)$$

Since

$$J_{3/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left(\frac{\sin x}{x} - \cos x\right), \quad (45)$$

we have

$$\mathcal{J}_m(q) = \left(\frac{2}{\pi k}\right)^{1/2} \left(\frac{1}{k} \int_0^\infty e^{-qr} r^{m-1} \sin(kr) dr - \int_0^\infty e^{-qr} r^m \cos(kr) dr\right). \quad (46)$$

The two integrals in equation (46) are elementary and involve Γ -functions. We readily find that, when $m > 0$,

$$\mathcal{J}_m(q) = -\left(\frac{2}{\pi k}\right)^{1/2} (m-1)! \rho^m k^{-1} \{m \rho k \cos[(m+1)\xi_0] - \sin(m\xi_0)\}, \quad (47)$$

where

$$\xi_0 = \tan^{-1} \frac{k}{q}, \quad \rho = (q^2 + k^2)^{-1/2}. \quad (48)$$

The cases $m = -1, m = 0$, occur in the evaluation of B_3 (eq. [44]). They can be treated separately to give

$$\mathcal{J}_{-1}(q) = \left(\frac{2}{\pi k}\right)^{1/2} \left(1 - \frac{q}{k} \xi_0\right); \quad \mathcal{J}_0(q) = \left(\frac{2}{\pi k}\right)^{1/2} \left(\frac{1}{k} \xi_0 - \rho \cos \xi_0\right). \quad (49)$$

It is now necessary only to use formulae (47), (48), and (49) to obtain the explicit forms of A , Γ_1 , Γ_2 , Γ_3 , Δ_1 , Δ_2 , B_1 , B_2 , B_3 , and B_4 . Combining these according to formulae (22), (28), (34), (36), (38), and (40) and rearranging the terms, we finally obtain

$$\mu_z = - (2048\pi^3)^{1/2} N \sigma^3 k^{-2} i \left\{ \begin{aligned} &6(\gamma + \epsilon + \zeta) \rho_3^4 (4\rho_3 k \cos 5\xi_1 \\ &- \sin 4\xi_1) + 2(\beta - 6\sigma\gamma + \delta + 6\sigma\epsilon) \rho_3^3 (3\rho_3 k \cos 4\xi_1 - \sin 3\xi_1) \\ &+ (1 + 12\sigma^2\gamma + 3\sigma\delta + 12\sigma^2\epsilon + 4\sigma^2\zeta) \rho_3^2 (2\rho_3 k \cos 3\xi_1 - \sin 2\xi_1) \\ &- 2\beta [\sigma\rho_4^2 (2\rho_4 k \cos 3\xi_2 - \sin 2\xi_2) + 6\sigma^2\rho_4 (\rho_4 k \cos 2\xi_2 - \sin \xi_2) \\ &+ 12\sigma^3 k \rho_4 \cos \xi_2 - 12\sigma^4 a (\xi_1 - \xi_2)] \end{aligned} \right\}, \quad (50)$$

where for brevity we have written

$$\left. \begin{aligned} \rho_3 &= (a^2 + k^2)^{-1/2}, & \xi_1 &= \tan^{-1} \frac{k}{a}, \\ \rho_4 &= [(2a+1)^2 + k^2]^{-1/2}, & \xi_2 &= \tan^{-1} \frac{k}{2a+1}, \end{aligned} \right\} \sigma = (a+1)^{-1}. \quad (51)$$

Finally the atomic absorption coefficient k_ν is given by equation (11), setting $|\mu| = |\mu_z|$. If we put $\delta = \epsilon = \zeta = 0$ in equation (50), we recover Massey and Bates's formula¹⁵ for the matrix element obtained by using the third-order wave-function for the ground state of H^- .

It may be of interest to see the numerical form which equation (11) for k_ν takes when the values of the constants from equation (7) are substituted in expressions (50) and (51):

$$k_\nu = 10^{-17} \times \frac{\nu_{at}}{k^3} \left\{ \begin{aligned} &1.7984 \rho_3^5 k \cos 5\xi_1 - 0.44961 \rho_3^4 \sin 4\xi_1 \\ &- 0.81823 \rho_3^4 k \cos 4\xi_1 + 0.27274 \rho_3^3 \sin 3\xi_1 + 2.3295 \rho_3^3 k \cos 3\xi_1 \\ &- 1.1648 \rho_3^2 \sin 2\xi_1 - 0.67760 \rho_3^3 k \cos 3\xi_2 + 0.33880 \rho_4^2 \sin 2\xi_2 \\ &- 1.1949 \rho_4^2 k \cos 2\xi_2 + 1.1949 \rho_4 \sin \xi_2 - 1.4048 \rho_4 k \cos \xi_2 \\ &+ 0.0101060 (\xi_1^\circ - \xi_2^\circ)^2 \text{cm}^2, \end{aligned} \right\} \quad (52)$$

where

$$\left\{ \begin{aligned} \nu_{at} &= \nu \times 2.4190 \times 10^{-17}; & k &= \sqrt{12.566 \nu_{at} - 0.052928}; & \tan \xi_1 &= 1.4261 k; \\ \tan \xi_2 &= 0.41625 k; & \rho_3 &= (0.49168 + k^2)^{-1/2}; & \rho_4 &= (5.7715 + k^2)^{-1/2}. \end{aligned} \right\} \quad (53)$$

The values for k_λ , as computed from equations (52) and (53), are given in Table 2. In Figure 1 we have illustrated the variation of the absorption coefficient with wave length and have compared it with the older absorption-coefficient-curve of Massey and Bates.

It will be seen that at the maximum the new absorption coefficient has a value about 15 per cent greater than the old and that, further, the position of maximum has been shifted by about 1000 Å to the red. A consequence of this is that the absorption coefficient now shows a smaller variation in the astronomical region of wave lengths than before; throughout the red and infrared regions the magnitude of the absorption is increased, and near the Paschen limit the increase is as much as 50 per cent.

¹⁵ *Op. cit.*, p. 206.

The changes of the form of the absorption coefficient are qualitatively such as would be expected to remove or diminish the discrepancy between the theoretical and observed

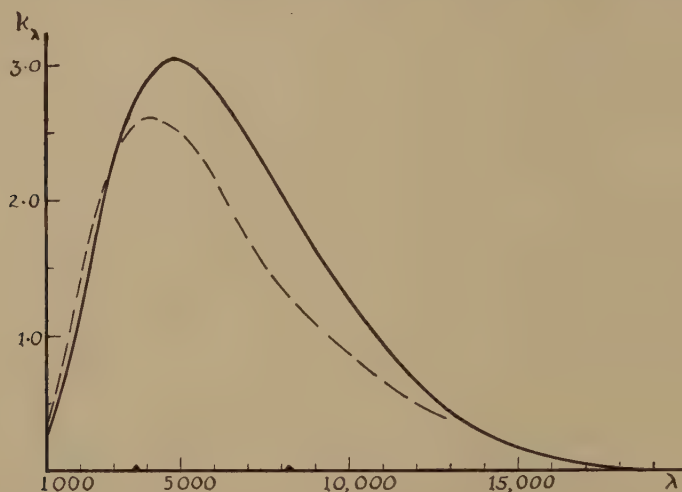


FIG. 1.—Absorption coefficient k_λ per negative hydrogen ion in units of 10^{-17} cm^2 , as a function of wave length in angstroms. The full-line curve is that obtained in the present paper; the broken line represents the previous calculation of Massey and Bates. The positions of the Paschen and the Balmer discontinuities are marked on the wave-length scale.

TABLE 2
CALCULATED ABSORPTION COEFFICIENT FOR H^-

λA	$k_\lambda \times 10^{17} \text{ cm}^2$	λA	$k_\lambda \times 10^{17} \text{ cm}^2$	λA	$k_\lambda \times 10^{17} \text{ cm}^2$	λA	$k_\lambda \times 10^{17} \text{ cm}^2$
1000.....	0.2733	4000.....	2.898	5750.....	2.899	9000.....	1.629
1500.....	0.6546	4250.....	2.973	6000.....	2.824	9500.....	1.436
2000.....	1.188	4500.....	3.017	6500.....	2.657	10,000.....	1.255
2500.....	1.761	4750.....	3.035	7000.....	2.461	12,000.....	0.6628
3000.....	2.268	5000.....	3.029	7500.....	2.253	14,000.....	0.2729
3250.....	2.477	5250.....	3.003	8000.....	2.042	16,000.....	0.07937
3500.....	2.653	5500.....	2.955	8500.....	1.832	17,217.....	0.00000
3750.....	2.792						

effective temperature-color-temperature relations. Calculations along these lines are now being undertaken to determine the extent to which the new absorption coefficient will modify current astrophysical theories.

The writer is deeply indebted to Dr. S. Chandrasekhar for his constant encouragement and friendly advice, without which this work would have been impossible.

YERKES OBSERVATORY
August 27, 1942

APPENDIX

THE CHARGE DISTRIBUTION OF THE H^- ION

The wave-function $\Psi(r_1, r_2; \vartheta_1, \vartheta_2; \varphi_1, \varphi_2)$ for any two-electron system implies a charge distribution about the nucleus given by

$$-\frac{dZ}{dr} = N^2 \int_{\vartheta_1=0}^{\pi} \int_{\varphi_1=0}^{2\pi} \int_{r_2=0}^{\infty} \int_{\vartheta_2=0}^{\pi} \int_{\varphi_2=0}^{2\pi} \Psi^2(r, r_2; \vartheta_1, \vartheta_2; \varphi_1, \varphi_2) r^2 r_2^2 dr_2 \sin \vartheta_1 \times \sin \vartheta_2 d\vartheta_1 d\vartheta_2 d\varphi_1 d\varphi_2 + N^2 \int_{r_1=0}^{\infty} \int_{\vartheta_1=0}^{\pi} \int_{\varphi_1=0}^{2\pi} \int_{r_2=0}^{\pi} \int_{\varphi_2=0}^{2\pi} \Psi^2(r_1, r; \vartheta_1, \vartheta_2; \varphi_1, \varphi_2) \times r_1^2 r^2 dr_1 \sin \vartheta_1 \sin \vartheta_2 d\vartheta_1 d\vartheta_2 d\varphi_1 d\varphi_2, \quad (1')$$

where N is the normalizing factor for Ψ . For a wave-function Ψ , which is symmetrical in the co-ordinates of the two electrons (such as the sixth-order function [6]), this becomes

$$-\frac{dZ}{dr} = 2N^2 \int_{\vartheta_1=0}^{\pi} \int_{\varphi_1=0}^{2\pi} \int_{r_2=0}^{\infty} \int_{\vartheta_2=0}^{\pi} \int_{\varphi_2=0}^{2\pi} \Psi^2(r, r_2; \vartheta_1, \vartheta_2; \varphi_1, \varphi_2) r^2 r_2^2 dr_2 \sin \vartheta_1 \times \sin \vartheta_2 d\vartheta_1 d\vartheta_2 d\varphi_1 d\varphi_2. \quad (2')$$

The charge distribution for the H^- ion can be obtained by substituting the wave-function (6) in equation (2'). The integrations may be performed by using the general methods of section III. This gives the expression

$$-\frac{dZ}{dr} = 32\pi^2 N^2 e^{-2ar} \left\{ r \cdot 2\beta \left[\frac{15(\epsilon + \gamma) + 9\zeta}{8a^7} + \frac{5\delta}{8a^6} + \frac{1}{4a^5} \right] (1 - e^{-2ar}) + r^2 \left(\frac{1}{4a^3} + \frac{3\delta}{4a^4} + \frac{3}{4a^5} [\beta^2 + \delta^2 + 2(\gamma + \epsilon + \zeta)] + \frac{15\delta}{4a^6} (\gamma + \epsilon + \zeta) + \frac{45}{8a^7} (\gamma + \epsilon + \zeta)^2 + 2\beta \left[\frac{5(\epsilon - \gamma)}{4a^6} + \frac{\delta}{4a^5} - e^{2ar} \left(\frac{25\epsilon + 5\gamma + 3\zeta}{8a^6} + \frac{3\delta}{4a^5} + \frac{1}{8a^4} \right) \right] \right) + r^3 \left(\frac{\delta}{2a^3} + \frac{3}{4a^4} (\delta^2 + 2[\epsilon - \gamma]) + \frac{3\delta}{2a^5} (3\epsilon - \gamma + \zeta) + \frac{15}{2a^6} (\gamma + \epsilon + \zeta)(\epsilon - \gamma) + 2\beta \left[\frac{2\epsilon + 2\gamma + 3\zeta}{2a^5} + \frac{3\delta}{8a^4} + \frac{1}{4a^3} - e^{-2ar} \left(\frac{2\epsilon}{a^5} + \frac{\delta}{4a^4} \right) \right] \right) + r^4 \left(\frac{1}{4a^3} [\beta^2 + \delta^2 + 2(\gamma + \epsilon + \zeta)] + \frac{3\delta}{4a^4} (3\epsilon + \zeta - \gamma) + \frac{3}{2a^5} (3\gamma^2 + 3\epsilon^2 + \zeta^2 - 2\epsilon\gamma + 2\gamma\zeta + 2\epsilon\zeta) + \frac{\zeta^2}{a^5} + 2\beta \left[\frac{3(\epsilon - \gamma)}{4a^4} + \frac{\delta}{4a^3} - \frac{\epsilon}{2a^4} e^{-2ar} \right] + r^5 \left[\frac{\delta}{2a^3} (\gamma + \epsilon + \zeta) + \frac{3}{2a^4} (\gamma + \epsilon + \zeta)(\epsilon - \gamma) + 2\beta \left(\frac{\gamma + \epsilon + \zeta}{4a^3} \right) + r^6 \left[\frac{1}{4a^3} (\gamma + \epsilon + \zeta)^2 \right] \right] \right\}. \quad (3')$$

Numerically, this is equivalent to (using the values of the constants for H^- from equations [7] and [9])

$$-\frac{dZ}{dr} = e^{-2ar}(2.087r + 1.362r^2 - 0.5418r^3 + 0.2614r^4 - 0.02952r^5 + 0.005734r^6) - e^{-4ar}(2.087r + 0.3118r^2 - 0.08749r^3 + 0.008112r^4).$$

(4')

TABLE 3

<i>r</i>	$-dZ/dr$	<i>r</i>	$-dZ/dr$	<i>r</i>	$-dZ/dr$	<i>r</i>	$-dZ/dr$
0.00.....	0.000	0.75.....	0.553	1.50.....	0.622	4.00.....	0.204
0.25.....	0.154	1.00.....	0.633	2.00.....	0.520	5.00.....	0.124
0.50.....	0.387	1.25.....	0.648	3.00.....	0.327	6.00.....	0.071

The charge distribution computed from expression (4') is shown in Table 3. Comparison with Bethe's calculation of the charge distribution,¹⁶ using his third-order wave-function, shows that the difference between his and the present one is for most purposes negligible.

¹⁶ *Zs. f. Phys.*, 57, 815, 1929.

THE CONTINUOUS SPECTRUM OF MODEL STELLAR ATMOSPHERES

RALPH E. WILLIAMSON

ABSTRACT

On the assumption that atomic hydrogen and the negative hydrogen ion are the only sources of continuous opacity in stellar atmospheres, the properties of the continuous spectra of twelve model atmospheres characterized by various values of the effective temperature T_e and the average electron pressure p_e , are investigated. An improved value of the atomic continuous absorption coefficient for H^- is used in the computations. Approximate values for the Rosseland mean absorption coefficients $\bar{\kappa}$ are derived for a wide range of the parameters p_e and T_e and for the twelve model atmospheres (with $10 \leq p_e \leq 10^4$; $5600^\circ \leq T_e \leq 10,080^\circ$); the monochromatic mass-absorption coefficients κ_ν are calculated for a number of selected frequencies covering a wave-length range of 2500–10,000 Å. The ratio $\bar{\kappa}/\kappa_\nu$ is used to determine the logarithm of the ratio of the monochromatic flux F_ν through each model atmosphere to the black-body flux $F_\nu^{(e)}$ corresponding to T_e for the atmosphere. Color temperatures for regions 3300–3650 Å, 3850–4550 Å, and 5550–7150 Å, as well as the magnitudes of the Balmer and the Paschen discontinuities in the continuous spectrum, are computed for each of the model atmospheres. Comparison of the calculated values of the Balmer discontinuity with observations of this quantity indicate a value of $p_e = 10^3$ for stars of spectral types A and a somewhat smaller value for stars of type F. Plots of the reciprocal color temperature $\Theta_e = 5040/T_e$ in three spectral regions, as a function of $\Theta_e = 5040/T_e$ for observations on stars, for the models here considered, and for those of Wildt, who used older values of the absorption coefficient of H^- (both the latter for $p_e = 10^3$), show that the agreement of theory and observation has been improved by the present calculations. In general, the agreement with the observations is less satisfactory in the longer wave lengths. It is shown that the present models predict some *increase* of T_e with frequency for the visual and near infrared, while the observations of Williams and of Hall show a slight *decrease* of T_e with frequency. The magnitude of the Paschen discontinuity for Vega as observed by Hall and Williams is satisfactorily predicted by an appropriate model ($\Theta_e = 0.5$, $p_e = 10^3$).

I. INTRODUCTION

An important problem of theoretical astrophysics is the prediction from purely physical principles of the radiant flux F_ν emerging from the surface of a star. While an ideal treatment of the problem would involve an accurate computation of F_ν (including the rapid variations in it which will constitute the absorption lines) for a number of model atmospheres defined by certain values of the parameters T_e , g , and the chemical abundances, it appears that in practice we can formally distinguish between a problem of the continuous spectrum of a star and a problem of the absorption lines.

On this understanding we suppose that we can consider the *continuous* spectrum of a star as arising from sources of opacity which change only slowly with frequency and can treat absorption lines as superposed on this background in such a way that the original large-scale distribution of energy is unaffected by them.

The calculation of the energy distribution in the continuous spectrum of a model stellar atmosphere of specified constitution was first undertaken by McCrea.¹ He investigated atmospheres composed of a mixture in different proportions of hydrogen and a hydrogen-like element with an ionization potential of 5 volts and compared the calculated continuous spectra with the observations of Yü. Since then similar calculations have been made, assuming various chemical compositions of the model atmospheres, by Unsöld² and by Pannekoek.³ However, Wildt's discovery⁴ of the importance of the negative hydrogen ion as a source of opacity in stellar atmospheres has necessitated a revision

¹ *M.N.*, **91**, 836, 1931.

² *Physik der Sternatmosphären*, Berlin: Julius Springer, 1938.

³ *Pub. Astronomical Institute of U. of Amsterdam*, No. 4.

⁴ *Ap.J.*, **89**, 295, 1939.

of these calculations. For the continuous spectrum this has been done by Wildt,⁵ while Strömgren⁶ has incorporated it into his treatment of the line spectrum of the sun. Wildt's work was based on values of the continuous absorption coefficient for H^- as computed by Massey and Bates,⁷ who used an approximate wave-function for the ion derived by Bethe⁸ and by Hylleraas.⁹ It has been recently found by the writer¹⁰ that a more accurate wave-function for H^- yields an absorption coefficient substantially different from that obtained by Massey and Bates. It is therefore of some interest to see in what way the change in the absorption coefficient will affect the calculations of Wildt on the continuous spectra of model stellar atmospheres. This is the principal object of this paper. More particularly, we shall investigate the emergent flux F_ν of the radiation in the continuous spectra of model stellar atmospheres, whose continuous opacity is supplied entirely by atomic hydrogen and negative hydrogen ions.¹¹ In the calculations the customary assumptions of local thermodynamic equilibrium and of the constancy of the ratio $\kappa_\nu/\bar{\kappa}$ throughout the atmosphere have been made. Because of the latter assumption it is convenient to use as the independent parameters the effective temperature T_e and the electron pressure p_e ¹² (instead of g).

II. THE ROSSELAND MEAN ABSORPTION COEFFICIENT

It is well known from the classical investigations on radiative equilibrium of stellar atmospheres by Milne and others¹³ that the emergent flux F_ν is given by

$$F_\nu = \frac{2h\nu^3}{c^2} \int_0^\infty \frac{K_2\left(\tau \cdot \frac{\kappa_\nu}{\bar{\kappa}}\right)}{e^{h\nu/kT_e\{3/4[\tau+q(\tau)]\}^{-1/4}} - 1} \frac{\kappa_\nu}{\bar{\kappa}} d\tau, \quad (1)$$

where κ_ν and $\bar{\kappa}$ denote the monochromatic and the Rosseland mean mass-absorption coefficients, respectively. Further, in equation (1) τ denotes the optical depth derived from κ and $q(\tau)$ is a slowly varying function of τ and approximately equal to $\frac{2}{3}$. Finally,

$$K_2(x) = \int_1^\infty e^{-tx} \frac{dt}{t^2}. \quad (2)$$

Burkhardt¹⁴ has evaluated equation (1) numerically and gives a table of the quantity $\log_{10} F_\nu/F_\nu^{(e)}$ as a function of the two parameters $h\nu/kT_e$ and $\bar{\kappa}/\kappa_\nu$, where

$$F_\nu^{(e)} = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT_e} - 1}. \quad (3)$$

⁵ *Ap.J.*, **90**, 611, 1939; **93**, 47, 1941.

⁶ *Festschrift für Elis Strömgren*, p. 218, Copenhagen, 1940.

⁷ *Ap.J.*, **91**, 202, 1940.

⁹ *Zs. f. Phys.*, **60**, 624, 1930.

⁸ *Zs. f. Phys.*, **57**, 815, 1929.

¹⁰ *Ap.J.*, **96**, 438, 1942.

¹¹ For a discussion of the validity of this assumption cf. Wildt, *Ap.J.*, **90**, 611, 1939.

¹² Thus the presence of free electrons from other sources than hydrogen is assumed, but no calculations based on abundances and ionizations of the "metals" are considered. Such calculations more properly belong in the theory of the formation of absorption lines (see Strömgren).

¹³ Cf. *Handbuch der Astrophysik*, **3**, Part III, 130, 147-55, Berlin: Julius Springer, 1930.

¹⁴ *Zs. f. Ap.*, **13**, 56, 1936.

It is therefore necessary to have a knowledge of both κ_ν and $\bar{\kappa}$ for each pair of values of T_e and p_e for which we desire the emergent flux F_ν . Since the run of F_ν with frequency is insensitive to small errors in κ and since a direct evaluation of $\bar{\kappa}$ would be quite tedious, the following approximate method may provide sufficient accuracy for our present purposes.

Let the Rosseland mean of the mass-absorption coefficient of atomic hydrogen alone be $\kappa(H)$ and that due to the negative hydrogen ion alone, $\bar{\kappa}(H^-)$. For most values of T_e and p_e one of the two mean absorption coefficients is much larger than the other. We can then take account of the presence of the less important source of opacity as a small correction to the more predominant one. Such a table of corrections has been provided by Strömgren¹⁵ in his work on the solar atmosphere. He tabulates the quantity $\log_{10} \bar{\kappa}/\kappa(H)$ as a function of $\log_{10} \bar{\kappa}(H^-)/\bar{\kappa}(H)$ and of T . It is further seen that the dependence on T of this correction term is practically negligible. The problem of obtaining the Rosseland mean $\bar{\kappa}$ is now greatly simplified, involving only the calculation of $\bar{\kappa}(H^-)$ and applying the appropriate correction term to the already well-known quantity $\bar{\kappa}(H)$.¹⁶

The contribution to the monochromatic absorption coefficient of H^- is made by bound-free and free-free transitions of the optical electron. The latter is comparatively unimportant except for wave lengths greater than 12,000 Å.

Let $a_\nu(H^-)$ denote the absorption coefficient per negative hydrogen ion due to the bound-free transitions; further, let $a_\nu^{(f)}(H^-)$ denote the absorption coefficient per neutral hydrogen atom and per unit electron pressure due to the free-free transitions. Then

$$\kappa_\nu(H^-) = \frac{1}{m_H} \left\{ \frac{N_{H^-}}{N_{H^-} + N_H + N_{H^+}} a_\nu(H^-) + \frac{N_H}{N_{H^-} + N_H + N_{H^+}} p_e a_\nu^{(f)}(H^-) \right\}. \quad (4)$$

N_x denotes the number of X -atoms (or X -ions) per unit volume, and m_H is the mass in grams of the hydrogen atom. Now

$$\left. \begin{aligned} \frac{N_H p_e}{N_{H^-}} &= 4 \frac{(2\pi m)^{3/2} (kT)^{5/2}}{h^3} e^{-h\nu_{H^-}/kT} = \phi(T), \\ \frac{N_{H^+} p_e}{N_H} &= \frac{(2\pi m)^{3/2} (kT)^{5/2}}{h^3} e^{-h\nu_H/kT} = \phi_0(T), \end{aligned} \right\} \quad (5)$$

where $h\nu_X$ is the first ionization energy of the X -atom or X -ion. Since $N_{H^-} \ll N_H$ for all cases of interest in stellar atmospheres, we may write equation (4) as

$$\kappa_\nu(H^-) = \frac{1}{m_H} \frac{p_e^2}{\phi_0 + p_e} \left\{ \frac{1}{\phi} a_\nu(H^-) + a_\nu^{(f)}(H^-) \right\}. \quad (6)$$

The effect of stimulated emissions (as negative absorptions) is readily taken into account. Denoting the "effective" absorption coefficient by κ'_ν , we have

$$\kappa'_\nu = \kappa_\nu (1 - e^{-h\nu/kT}). \quad (7)$$

¹⁵ *Op. cit.*, Table 4.

¹⁶ See, e.g., Wildt, *Aph.J.*, **90**, 611, Table 1, 1939.

Now

$$\overline{\kappa'}(H^-) = \frac{1}{m_H} \frac{p_e^2}{\phi_0 + p_e} \overline{a}(H^-), \quad (8)$$

where

$$[\overline{a}(H^-)]^{-1} = \frac{15}{4\pi^4} \int_0^\infty \frac{dB_\nu}{dT} \left\{ (1 - e^{-h\nu/kT}) \left[\frac{1}{\phi} a_\nu(H^-) + a_\nu^{(f)}(H^-) \right] \right\}^{-1} d\nu. \quad (9)$$

In future by "absorption coefficient" we shall always mean the "effective" absorption coefficient including the stimulated emissions as in equation (7); we shall accordingly suppress the primes. Since the integrand in equation (9) is independent of p_e , it is clear that one set of integrations for different values of T is all that is needed to obtain $\overline{\kappa}(H^-)$. Table 1 gives $\overline{a}(H^-)$ as computed by graphical integration, using the revised

TABLE 1
ROSSELAND MEAN OF H^- ABSORPTION COEFFICIENT/GRAM NEUTRAL H /DYNE p_e

$\Theta = \frac{5040}{T}$	$\overline{a}_H \times 10^{27}$ Present	$\overline{a}_H \times 10^{27}$ Wheeler and Wildt	$\Theta = \frac{5040}{T}$	$\overline{a}_H \times 10^{27}$ Present	$\overline{a}_H \times 10^{27}$ Wheeler and Wildt
0.5.....	2.72		1.4.....	26.8	27.7
0.6.....	5.80		1.6.....	26.2	27.7
0.8.....	14.5	12.4	1.8.....	24.8	25.5
1.0.....	21.8	19.4	2.0.....	23.1	23.9
1.2.....	25.7	25.7			

values of Williamson¹⁷ for $a_\nu(H^-)$ and the results of Wheeler and Wildt for $a_\nu^{(f)}(H^-)$.¹⁸ The last column of Table 1 gives for comparison the results of a similar computation by Wheeler and Wildt, who used the earlier values of the bound-free absorption coefficient according to Massey and Bates. It is seen that no large changes in $\overline{a}(H^-)$ have been introduced by the new calculations. By interpolating among the values given there and substituting into equation (8), we can readily obtain $\overline{\kappa}(H^-)$. In Table 2 the common logarithm of this quantity has been entered in the top line of each square; further in this table the second line contains $\log_{10} \overline{\kappa}(H)$ from Wildt's paper,¹⁶ while the third gives $\log_{10} \overline{\kappa}$, derived after applying Strömgren's corrections.

III. THE CONTINUOUS SPECTRUM OF THE MODEL ATMOSPHERES

As we have already stated in the introductory section, to determine the emergent flux F_ν for a given model stellar atmosphere, we need the value of the absorption coefficient at the particular frequency ν in which we are interested, as well as the Rosseland mean absorption coefficient κ . We have already provided the necessary table of κ in the pre-

¹⁷ *Loc. cit.*

¹⁸ *A p. J.*, 95, 281, 1942.

ceding section. It now remains to evaluate κ_ν , which is the sum of the absorption coefficient $\kappa_\nu(H^-)$ due to H^- and the absorption coefficient $\kappa_\nu(H)$ due to hydrogen. The function $\kappa_\nu(H^-)$ can be readily found from equation (4) and the table of monochromatic atomic continuous absorption coefficients provided by the author.¹⁹ For $\kappa_\nu(H)$ the customary Kramers' approximation was used as outlined, for example, in Unsöld's book. Once $\kappa_\nu = \kappa_\nu(H^-) + \kappa_\nu(H)$ and $\bar{\kappa}$ are both known, the ratio of the emergent flux F_ν to

TABLE 2*
ROSSELAND MEAN ABSORPTION COEFFICIENT/GRAM STELLAR MATERIAL

Θ_e	p_e							
	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3	10^4
0.5								1.19
	-3.97	-2.97	-1.97	-0.97	0.02	0.96	1.59	1.77
	-3.97	-2.97	-1.97	-0.97	0.02	0.96	1.59	1.89
0.6						-0.52	0.53	1.54
	-3.58	-2.58	-1.58	-0.61	0.21	0.57	0.62	0.63
	-3.58	-2.58	-1.58	-0.61	0.21	0.62	0.95	1.62
0.7					-1.23	-0.21	0.79	1.79
	-3.29	-2.30	-1.37	-0.78	-0.63	-0.61	-0.61	
	-3.29	-2.30	-1.37	-0.78	-0.49	0.00	0.81	1.79
0.8			-3.12	-2.07	-1.06	-0.06	0.94	1.94
	-3.06	-2.25	-1.90	-1.85	-1.84			
	-3.06	-2.25	-1.86	-1.58	-0.95	-0.06	0.94	1.94
0.9		-3.98	-2.96	-1.96	-0.96	0.04	1.04	2.04
	-3.27	-3.11	-3.09	-3.09				
	-3.27	-3.02	-2.65	-1.91	-0.96	0.04	1.04	2.04
1.0	-4.89	-3.89	-2.89	-1.89	-0.89	0.11	1.11	2.11
	-4.33	-4.33	-4.33					
	-4.18	-3.69	-2.88	-1.89	-0.89	0.11	1.11	2.11
1.1	-4.84	-3.84	-2.84	-1.84	-0.84	0.16	1.16	2.16
	-4.84	-3.84	-2.84	-1.84	-0.84	0.16	1.16	2.16
	-4.84	-3.84	-2.84	-1.84	-0.84	0.16	1.16	2.16
1.2	-4.81	-3.81	-2.81	-1.81	-0.81	0.19	1.19	2.19
	-4.81	-3.81	-2.81	-1.81	-0.81	0.19	1.19	2.19
	-4.81	-3.81	-2.81	-1.81	-0.81	0.19	1.19	2.19

* Each square contains: first line, $\log_{10} \bar{\kappa}(H^-)$; second line, $\log_{10} \bar{\kappa}(H)$; third line, $\log_{10} \bar{\kappa}$.

the flux $F_\nu^{(e)}$ (which is the black-body flux corresponding to the effective temperature T_e) can at once be obtained by interpolating in Burkhardt's table. Tables 3, 4, and 5 and Figures 1, 2, and 3 summarize the results of such computations.

In Table 3 the quantities $\kappa_\nu(H^-)$ per gram of neutral hydrogen and per bar of electron pressure and $\kappa_\nu(H)$ per gram of neutral hydrogen are tabulated for different values of the reciprocal effective temperature $\Theta_e = 5040/T_e$ and for a number of selected frequencies.

¹⁹ *Op. cit.*, Table 2.

In Table 4 the final monochromatic absorption coefficients per gram of stellar material are given for twelve different model atmospheres; Table 4 also contains the quantities $\log_{10} F_\nu/F_\nu^{(e)}$ for these models at the selected frequencies. At the Paschen and the

TABLE 3*
MONOCHROMATIC ABSORPTION COEFFICIENTS

$\nu \times 10^{-15}$	θ				
	0.5	0.6	0.7	0.8	0.9
0.30.....	1.62 100.6	2.73 4.646	4.48 0.2207	7.14 1.070×10^{-2}	11.09 5.279×10^{-4}
0.365.....	2.15 { 55.70 } { 171.6 }	3.85 { 2.573 } { 9.752 }	6.52 { 0.1222 } { 0.5665 }	10.60 { 5.927×10^{-3} } { 3.340×10^{-2} }	16.66 { 2.923×10^{-4} } { 1.993×10^{-3} }
0.400.....	2.44 130.8	4.40 7.430	7.51 0.4315	12.26 2.544×10^{-2}	19.32 1.518×10^{-3}
0.420.....	2.57 113.2	4.64 6.432	7.96 0.3736	13.00 2.203×10^{-2}	20.48 1.314×10^{-3}
0.540.....	3.04 53.26	5.60 3.026	9.66 0.1758	15.86 1.036×10^{-2}	25.07 6.185×10^{-4}
0.600.....	3.12 38.74	5.76 2.201	9.93 0.1278	16.32 7.540×10^{-3}	25.82 4.499×10^{-4}
0.660.....	3.10 29.18	5.72 1.658	9.88 9.628×10^{-2}	16.25 5.678×10^{-3}	25.71 3.388×10^{-4}
0.779.....	2.89 17.68	5.35 1.004	9.28 5.833×10^{-2}	15.26 3.440×10^{-3}	24.16 2.052×10^{-4}
0.800.....	2.84 16.34	5.27 0.9287	9.12 0.05394	15.00 3.181×10^{-3}	23.74 1.898×10^{-4}
0.822.....	2.79 { 15.07 } { 317.1 }	5.17 { 0.8562 } { 29.74 }	8.95 { 0.04973 } { 2.811 }	14.73 { 2.932×10^{-3} } { 0.2667 }	23.31 { 1.750×10^{-4} } { 2.539×10^{-2} }
0.899.....	2.55 242.1	4.74 22.71	8.22 2.146	13.55 0.2037	21.48 1.939×10^{-2}
1.00.....	2.30 176.1	4.27 16.52	7.40 1.561	12.18 0.1481	19.29 1.410×10^{-2}
1.40.....	1.36 64.18	2.53 6.020	4.39 0.5689	7.23 5.399×10^{-2}	11.46 5.139×10^{-3}

* Each square contains: first line, $\kappa_\nu (H^-)$ per gram of neutral H and for an electron pressure of 10^3 bars; second line, $\kappa_\nu (H)/\text{gm neutral } H$. The values in braces are those at either side of a discontinuity.

Balmer discontinuities the relevant quantities are given for both sides of the discontinuity.

Figure 1 shows a plot of the absorption coefficient for the model atmosphere for $\theta_e = 0.5$, $p_e = 10^3$ bar.

TABLE 4*

ABSORPTION COEFFICIENT AND FLUX IN MODEL ATMOSPHERES

$\nu \times 10^{-13}$	p_e											
	10^1			10^2			10^3			10^4		
	Θ_e											
	0.8	0.9	0.6	0.7	0.8	0.9	0.5	0.6	0.7	0.8	0.6	0.7
0.300.....	0.0736 + .036	0.103 - .007	3.51 +0.002	0.575 +0.047	0.651 +0.019	1.02 -0.008	50.4 - 0.043	5.96 + 0.026	4.06 + 0.038	6.42 + 0.021	26.2 + 0.035	38.9 + 0.038
0.365.....	{ .105 } { .131 } { .008 } { .042 }	{ .160 } { .161 } { .078 } { .081 }	{ 2.25 } { 7.72 } { + 0.058 } { - 0.086 }	{ 0.703 } { 1.11 } { +0.029 } { -0.033 }	{ 1.00 } { 1.03 } { -0.037 } { -0.044 }	{ 1.59 } { 1.60 } { -0.078 } { -0.078 }	{ 30.9 } { 92.9 } { + 0.009 } { - 0.095 }	{ 5.54 } { 11.7 } { + 0.041 } { - 0.053 }	{ 6.06 } { 6.46 } { - 0.007 } { - 0.016 }	{ 9.95 } { 9.97 } { - 0.037 } { - 0.037 }	{ 35.9 } { 42.3 } { 0.000 } { - 0.016 }	{ 59.6 } { 60.0 } { - 0.013 } { - 0.013 }
0.400.....	.141 - .055	.188 - .110	6.15 - 0.067	1.10 - 0.031	1.19 - 0.069	1.87 - 0.107	73.5 - 0.084	10.5 - 0.040	7.39 - 0.038	11.7 - 0.066	46.1 - 0.032	70.3 - 0.037
0.420.....	.146 - .061	.200 - .121	5.45 - 0.054	1.09 - 0.033	1.27 - 0.079	1.99 - 0.120	64.9 - 0.074	9.92 - 0.033	7.82 - 0.048	12.5 - 0.077	48.0 - 0.038	75.1 - 0.049
0.540.....	.166 - .093	.249 - .183	2.97 + 0.041	1.11 - 0.034	1.57 - 0.127	2.48 - 0.182	33.7 + 0.002	8.11 0.000	9.56 - 0.086	15.6 - 0.127	56.2 - 0.067	94.1 - 0.092
0.600.....	.169 - .101	.257 - .204	2.34 + 0.097	1.10 - 0.032	1.62 - 0.144	2.57 - 0.203	25.6 + 0.051	7.58 + 0.015	9.87 - 0.097	16.2 - 0.143	57.7 - 0.074	97.6 - 0.104
0.660.....	.167 - .103	.257 - .217	1.89	1.07 - 0.024	1.62 - 0.154	2.56 - 0.217	20.0 + 0.103	7.10 + 0.034	9.85 - 0.103	16.2 - 0.153	57.4 - 0.076	97.7 - 0.110
0.779.....	.155 - .084	.242 - .229	1.32	0.976 + 0.007	1.53 - 0.153	2.41 - 0.230	13.01 + 0.260..	6.19 + 0.084	9.28 - 0.095	15.2 - 0.153	53.8 - 0.066	92.3 - 0.104
0.800.....	.153 - .078	.237 - .227	1.25	0.957 + 0.016	1.50 - 0.149	2.37 - 0.227	12.2	6.04 + 0.093	9.12 - 0.089	15.0 - 0.149	53.0 - 0.062	90.8 - 0.101
0.822.....	{ .150 } { .413 } { .072 } { .336 }	{ .233 } { .258 } { .226 } { .256 }	{ 1.18 } { 26.1 } { + 0.42.. } { - 0.326 }	{ 0.937 } { 3.67 } { + 0.027 } { - 0.302 }	{ 1.47 } { 1.74 } { - 0.146 } { - 0.193 }	{ 2.33 } { 2.35 } { - 0.224 } { - 0.229 }	{ 11.4 } { 203. } { + 0.320.. } { - 0.259 }	{ 5.89 } { 34.1 } { + 0.104 } { - 0.267 }	{ 8.96 } { 11.7 } { - 0.085 } { - 0.155 }	{ 14.7 } { 15.0 } { - 0.144 } { - 0.151 }	{ 52.0 } { 80.6 } { - 0.059 } { - 0.156 }	{ 89.2 } { 91.9 } { - 0.097 } { - 0.105 }
0.899.....	.338 - .320	.234 - .238	20.0 - 0.323	2.95 - 0.281	1.56 - 0.168	2.17 - 0.209	156. - 0.254	26.9 - 0.250	10.3 - 0.128	13.7 - 0.125	69.6 - 0.131	84.2 - 0.080
1.00.....	.269 - .274	.207 - .195	14.7 - 0.304	2.29 - 0.239	1.37 - 0.123	1.94 - 0.167	115. - 0.234	20.4 - 0.215	8.95 - 0.084	12.3 - 0.083	59.0 - 0.091	75.5 - 0.039
1.40.....	.126 + 0.126	.120 + 0.210	5.45 - 0.045	1.00 + 0.128	0.777 + 0.273	1.15 + 0.240	42.4 + 0.003	8.42 + 0.105	4.96 + 0.284	7.29 + 0.316	31.3 + 0.215	44.5 + 0.318

* Each square contains: first line, κ_0 in cm^2/gm ; second line, $\log_{10} F_p/F_p\omega$. The values in braces are those at either side of a discontinuity.Values of the quantity $\log_{10} F_p/F_p\omega$, which have been omitted from the above table, correspond to values of ξ/ω , which lie outside the range of Burkhart's table. In a few instances approximate values for $\log_{10} F_p/F_p\omega$ have been arrived at by extrapolation of Burkhart's data, and these have been marked by the symbol ...

In Figure 2 we have plotted the values of $\log_{10} F_\nu/F_\nu^{(e)}$ as a function of frequency for the various models. For practical purposes it is convenient to have values of $\log_{10} F_\nu/F_\nu^{(e)}$ because it is directly comparable with the observations for the gradients made on the continuous spectra of stars and also because it is closely related to color temperatures. On such a plot as Figure 2, energy distributions corresponding to any color temperature are approximately straight lines. Thus index marks made in this figure show the slopes corresponding to various reciprocal color temperatures.

Figure 3 gives the distribution of the emergent flux in absolute units as a function of frequency for two of the model atmospheres considered: $\Theta_e = 0.5$, $p_e = 10^3$; and $\Theta_e = 0.8$, $p_e = 10^3$.

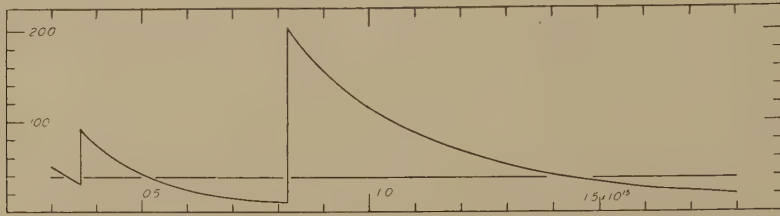


FIG. 1.—The monochromatic absorption coefficient κ_ν for a model stellar atmosphere with $\Theta_e = 5040/T_e = 0.5$, $p_e = 10^3$ bar. The ordinates denote κ_ν in units of cm^2 per gram, while the abscissae denote frequencies in cycles per second, one small division representing 0.1×10^{15} cycles. The ordinate of the horizontal line corresponds to the value of κ .

Table 5 gives computed color temperatures for the models at various frequency ranges of astrophysical interest, as well as the logarithm of the ratio of the fluxes on either side of the Balmer and the Paschen discontinuities.

IV. COMPARISON OF THE RESULTS FOR MODEL STELLAR ATMOSPHERES WITH THE DATA OF OBSERVATIONS

An important feature of the emergent flux of the radiation of stellar atmospheres is the sudden discontinuity at the head of the Balmer series. Since the model atmospheres show large variations in the amount of this discontinuity with the electron pressure, it is natural to suppose that for a given effective temperature the average electron pressure in a stellar atmosphere will be indicated by the observed magnitude of the Balmer discontinuity. Now Barbier and Chalonge²⁰ give as the final averages of their measures of the quantity $D = \Delta \log_{10} F_\nu$ at the Balmer discontinuity, for more than 200 main-sequence stars, the following values: A0, 0.47; A2, 0.44; A3, 0.42; A5, 0.39; F0, 0.28; F2, 0.22; F5, 0.17; F8, 0.11. Using Kuiper's temperature scale,²¹ we can refer these values for the different spectral types to appropriate effective temperatures; and in this manner we can compare the observed values of D with those given in Table 5, calculated for the model atmospheres. It is found that with $p_e = 10^3$ we obtain values of D agreeing with those observed in the earlier spectral types, while a slight decrease of p_e is indicated for the later ones.

Barbier and Chalonge²² have also measured stellar color temperatures on either side of the Balmer discontinuity. The spectral ranges of their measures correspond closely with those of the computed color temperatures B and C of Table 5. Figure 4 contains a

²⁰ *C.R.*, 210, 99, 1940.

²¹ *Ap.J.*, 88, 429, 1938.

²² *Op. cit.*

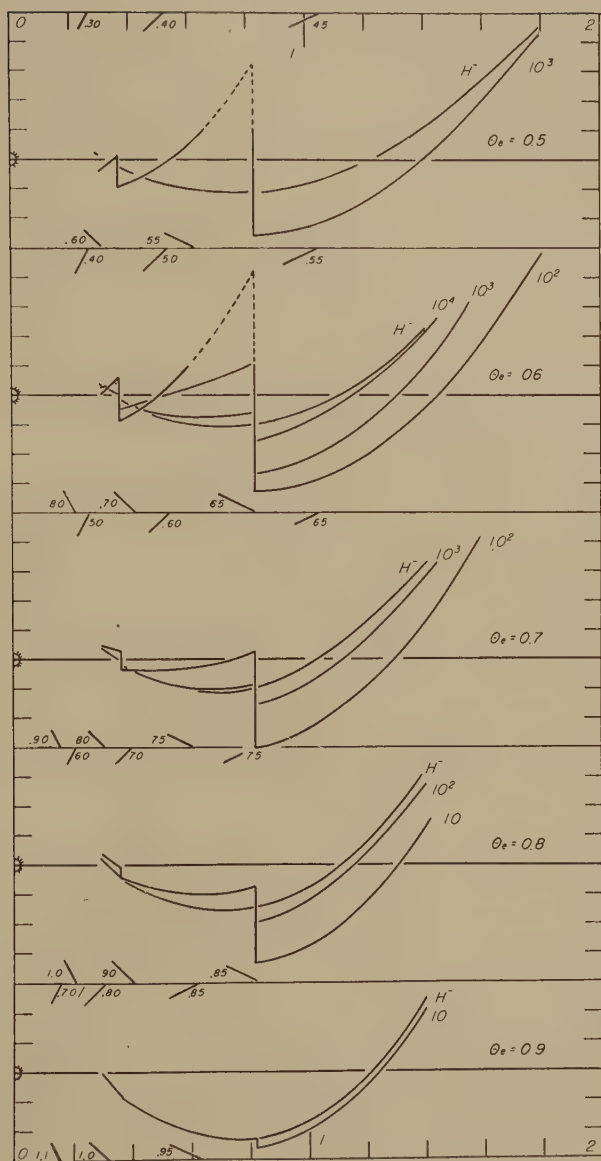


FIG. 2.—The deviations of the emergent flux F_ν of model stellar atmospheres from the radiation $F_\nu^{(e)}$ from a black surface at the effective temperature. The ordinates represent $\log_{10} F_\nu / F_\nu^{(e)}$ (one small division corresponding to 0.1 in the logarithm), while the abscissae represent frequencies (one small division corresponding to a frequency difference of 0.2×10^{15} cycles per second). The numbers labeling the curves correspond to the electron pressures of the model atmospheres which they represent. The curves marked " H^- " correspond to the results for atmospheres of pure H^- . Index marks give the approximate slopes corresponding to the various values of θ_e .

plot of the reciprocal color temperatures as a function of the reciprocal effective temperatures: (1) of Barbier and Chalonge's observations; (2) of Wildt's²³ earlier calculations for $p_e = 10^3$; and (3) for the model atmospheres computed in this paper, also for $p_e = 10^3$. In both cases the present models give a slope nearer that of the observed stars. It

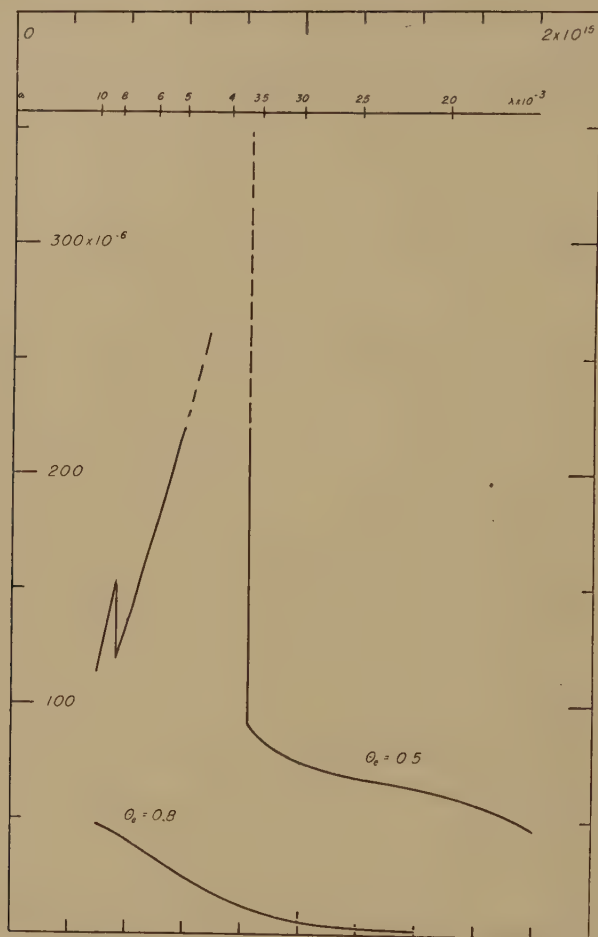


FIG. 3.—The frequency dependence of the emergent flux for two model stellar atmospheres ($\Theta_e = 0.5$, $p_e = 10^3$; $\Theta_e = 0.8$, $p_e = 10^3$). The quantity F_ν (in cgs units) is plotted along the ordinate, while one small division of the abscissa corresponds to a frequency difference of 0.2×10^{15} cycles per second. An auxiliary scale represents the wave length in thousands of angstroms.

should be noted that the point with the largest value of Θ_e is for the sun, measured by Arnulf, Chalonge, and Jardin,²⁴ and is doubtless not directly comparable with the other observations. This is because of the fact that they may be based on slightly different zero points and also because the dispersions were different, thus introducing a variable influence of absorption lines on the gradients. Since the zero point of the Barbier and

²³ *Ap.J.*, 93, 47, 1941.

²⁴ *C.R.*, 210, 325, 1940.

Chalonge color observations may be uncertain, the vertical placing of the observed points in Figure 4 may be subject to uncertainties. However, it appears that the differences between the reciprocal color temperatures for the two regions should be less subject to errors of the zero point than are the quantities themselves. Accordingly, in Figure 5 we have compared observed values of the differences in the color temperatures at $\lambda\lambda$ 4600–3700 and $\lambda\lambda$ 3700–3150 with the results for the model atmospheres. It is seen that the

TABLE 5*
THEORETICAL RECIPROCAL COLOR TEMPERATURES

Θ_e	0.5	0.6	0.7	0.8	0.9
$p_e=10$					
A.....				0.86	1.02
B.....				.76	0.92
C.....				.75	0.84
D.....				.264	0.030
E.....				0.034	0.003
$p_e=10^2$					
A.....		0.42	0.70	0.89	1.02
B.....			.64	.80	0.93
C.....		.59	.63	.72	0.84
D.....		.75::	.329	.047	0.005
E.....		0.144	0.062	0.007	0.000
$p_e=10^3$					
A.....	0.36	0.54	0.77	0.90	
B.....	.31::	.50	.68	.80	
C.....	.48	.55	.61	.72	
D.....	.58::	.371	.070	.007	
E.....	0.104	0.094	0.009	0.000	
$p_e=10^4$					
A.....		0.65	0.78		
B.....		.58	.69		
C.....		.52	.62		
D.....		.097	.008		
E.....		0.016	0.000		
Limit					
{A.....	0.57	0.67	0.79	0.90	1.02
{B.....	.51	.61	.68	.80	0.93
{C.....	0.48	0.54	0.63	0.73	0.83
p_e^*	10^7	10^6	10^4	10^3	10^2

The bottom row gives limiting values of the color temperatures for stellar atmospheres, where only the opacity due to H^- is effective. The electron pressure, p_e^* , at which this occurs, is also shown for each temperature.

* $\Theta_e = 5040/T_e = \begin{matrix} A, & B, & C, \\ 10^4/\lambda = 1.40-1.80, & 2.20-2.60, & 2.74-3.00, \\ D = \Delta \log_{10} F_\nu & \text{at Balmer limit, } E = \Delta \log_{10} F_\nu & \text{at Paschen limit.} \end{matrix}$

computed relations are not in exact agreement with the observed ones, but it is also seen that our present calculations represent a slight improvement over Wildt's calculation.²⁵

Now the Greenwich observers have determined the color temperatures of a total of more than 300 stars,²⁶ in a wave-length region between 4100 Å and 6500 Å. This is a range which is nearly equal to the extremes of the ranges of the computed reciprocal color

²⁵ The fact that the point for the sun is in such marked disagreement with the run of the other observed values seems to indicate that the solar observation is not directly comparable with the others.

²⁶ Sir Frank Dyson, *Observations of Color-Temperatures of Stars, 1926-1932*, London, 1932; also *M.N.*, 100, 189, 1940.

temperatures A and B (see Table 5). Thus an approximate value of the reciprocal color temperature of the model atmospheres throughout the whole region may be taken to be the average of A and B . Figure 6 shows the relation between the reciprocal effective temperatures and the reciprocal color temperatures for this spectral region for (1) averages of stars of various spectral types as observed at Greenwich; (2) Wildt's model atmospheres; and (3) the model atmospheres computed here (both [2] and [3] corresponding to $p_e = 10^3$). A value of $\phi_0 = 1.00$ was taken for the Greenwich gradients; and so the whole set of observed points in Figure 6 may be subject to a slight displacement in the vertical direction. However, the general disagreement of slopes could not be re-

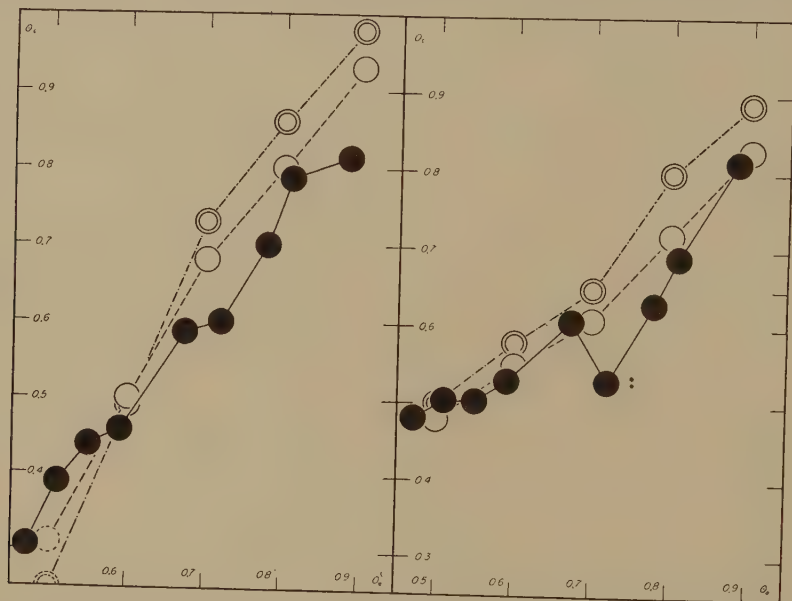


FIG. 4.— $(\Theta_c - \Theta_e)$ relations as observed and computed, for the intervals $\lambda\lambda$ 4600–3700 (the diagram to the left) and $\lambda\lambda$ 3700–3150 (the diagram to the right). The solid dots represent observations of Barbier and Chalonge. The single circles represent theoretical values for model atmospheres, with $p_e = 10^3$; and the double circles correspond to the earlier calculations of Wildt (*A p. J.*, 93, 47, 1941). The observed point at $\Theta_e = 0.72$ is quite uncertain (cf. Barbier and Chalonge, *Ann. d'ap.*, Vol. 3, No. 2, 1940). The point at $\Theta_e = 0.89$ is for the sun; and, since it was obtained in a different manner from the others, is not strictly comparable with them.

moved by any such translation. It is obvious that the agreement of the model atmospheres with the visual (Greenwich) gradients is less satisfactory than with either the photographic or the ultraviolet gradients (Barbier and Chalonge). However, the disagreements, such as they are, are all in the same sense: the $\Theta_c - \Theta_e$ relation is steeper for the model atmospheres than that which the observations indicate. Altogether, the more accurate H^- absorption coefficient which has been used in the present calculations seems to have decreased—definitely, if not sufficiently—the disagreement between the model atmospheres and the true stellar atmospheres. Wildt has pointed out²⁷ already that this discrepancy cannot be traced to the effect of absorption lines, since it operates in the opposite direction.

The model atmospheres show a discontinuity in F_ν at the head of the Paschen series similar to that at the head of the Balmer series, although considerably smaller. Hall and Williams²⁸ have measured the quantity $D_P = \Delta \log_{10} F_\nu$ at the Paschen discontinuity for

²⁷ *A p. J.*, 93, 47, 1941.

²⁸ *A p. J.*, 95, 225, 1942.

the A0 star Vega and find it to be approximately 0.092. Reference to Table 5 shows that for a model atmosphere with $\Theta_e = 0.5$ and $p_e = 10^3$ the quantity $D_P = 0.104$, which is in satisfactory agreement with the estimate of Hall and Williams. However, it should be remarked that Hall and Williams give their value as a minimum and state that further observations will be necessary to establish the exact magnitude of the discontinuity.

Accurate investigations of the linearity of stellar gradients in the visual and the near infrared regions have been made recently by Williams²⁹ and by Hall,³⁰ respectively. Williams finds only slight systematic deviations from linearity in the region $\lambda\lambda$ 4000–6700, for stars of types A and F. The small differences in gradient which he does find

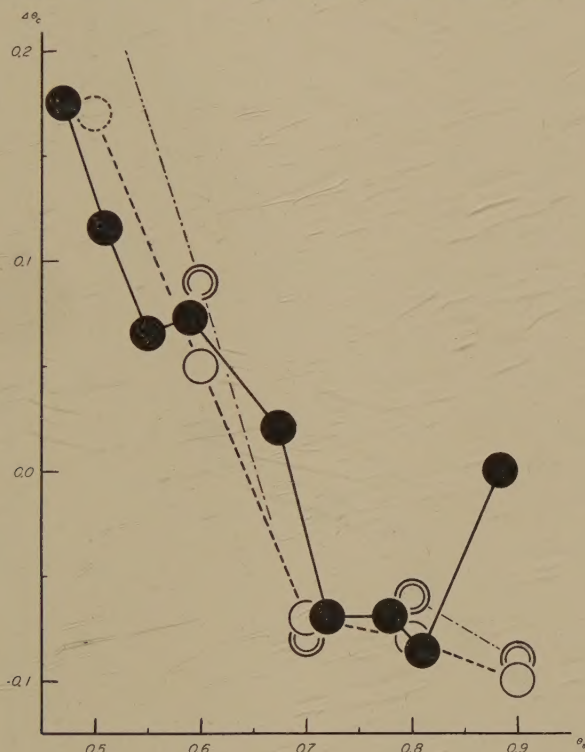


FIG. 5.— $\Delta\Theta_e = \Theta_e(\lambda\lambda\ 3700\text{--}3150) - \Theta_e(\lambda\lambda\ 4600\text{--}3700)$ as observed and as calculated. The ordinates denote $\Delta\Theta_e$, and the abscissae denote Θ_e . The other symbols have the same meaning as in Fig. 4.

make the color temperatures higher in the red than in the blue. Reference to Figure 2 shows that this curvature is opposite in sign to that exhibited by the model atmospheres. Hall's results for the near infrared ($\lambda\lambda$ 4600–10,000) support the assumption that the main-sequence stars of types A and F radiate essentially as black bodies in this region of the spectrum, except for the Paschen discontinuity. On the scale of Figure 2 the curvature found by either Hall or Williams would not be noticeable over the ranges of their respective observations.

To summarize: The model atmospheres studied in this paper, with the help of the continuous absorption coefficient of H^- as recently recomputed, predict an effective temperature to color temperature relation which fits observed curves fairly satisfactorily in the ultraviolet, less so in the photographic region, and still less so in the visual region.

²⁹ *Pub. Obs. U. Michigan*, **7**, 93, 147, 159, 1939.

³⁰ *Ap.J.*, **95**, 231, 1942.

In all three cases, however, the theoretical relations derived here lie somewhat closer to the observational ones than those obtained earlier by Wildt, who used a less accurate absorption coefficient for H^- .

The models treated in the present paper show a definite trend of the color temperature to increase with increasing frequency, while actual stars seem to show a fairly constant color temperature, which, if anything, decreases slightly with frequency.

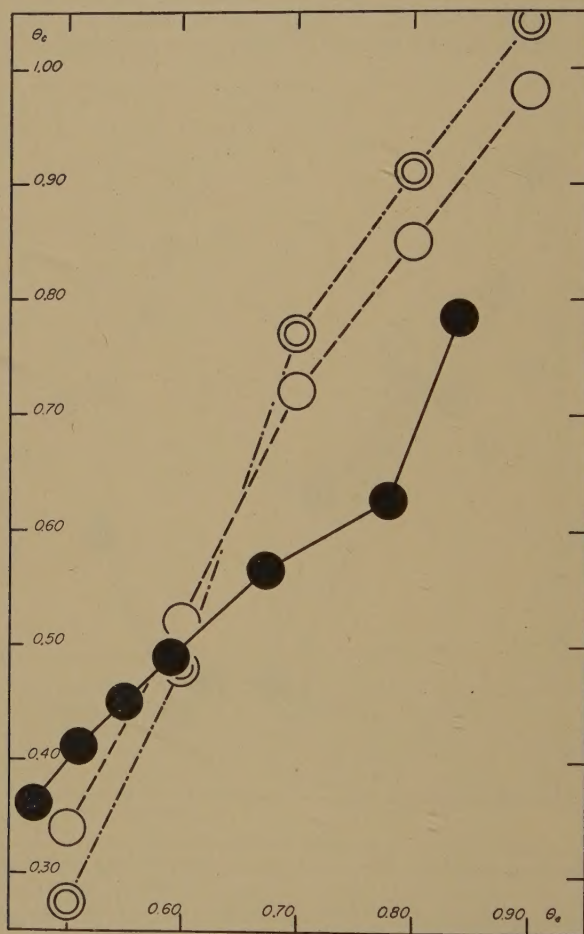


FIG. 6.—The $(\Theta_c - \Theta_e)$ relation in the spectral region $\lambda\lambda$ 6500–4100, as observed and as calculated. The filled circles represent the Greenwich observations. The single circles correspond to the values calculated in the present paper with $p_e = 10^3$, while the double circles represent the values according to Wildt's earlier calculation.

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